

NEW SUPPLEMENTARY CONDITIONS FOR A NON-LINEAR FIELD THEORY: GENERAL RELATIVITY

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Abstract. The Einstein theory of general relativity provides a peculiar example of classical field theory ruled by non-linear partial differential equations. A number of supplementary conditions (more frequently called gauge conditions) have also been considered in the literature. In the present paper, starting from the de Donder gauge, which is not conformally invariant but is the gravitational counterpart of the Lorenz gauge, we consider, led by geometric structures on vector bundles, a new family of gauges in general relativity, which involve fifth-order covariant derivatives of metric perturbations. A review of recent results by the authors is presented: restrictions on the general form of the metric on the vector bundle of symmetric rank-two tensor fields over space-time; admissibility of such gauges in the case of linearized theory about flat Euclidean space; generalization to a suitable class of curved Riemannian backgrounds, by solving an integral equation. Eventually, the applications to Euclidean quantum gravity are discussed.

In the analysis of classical field theories which rely on partial differential equations, the Einstein theory of general relativity has a distinctive feature because it describes the gravitational field as a non-linear system even in the absence of other fields.¹ The self-interaction of the gravitational field occurs because the space-time over which it propagates is defined by gravity itself. Solutions of the Einstein field equations can be unique only up to a diffeomorphism, and a fixed background metric is introduced to obtain a definite member of the equivalence class of metrics which represents a space-time. For this purpose, one has also to impose four supplementary conditions on the covariant derivatives of the physical metric with respect to the background metric. The four degrees of freedom to make diffeomorphisms are hence removed, and a unique solution for the metric components is obtained. Moreover, since the metric defines the space-time structure, one does not know in advance what the region is on which the solution should be determined.¹ All what one has is a three-manifold Σ with certain initial data \mathcal{I} on it, and one has to find a four-manifold M , an imbedding

$$\theta : \Sigma \rightarrow M$$

and a metric g on M which satisfies the Einstein equations (T_{ab} denotes the energy-momentum tensor)

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi GT_{ab}, \quad (1)$$

agrees with the initial values on $\theta(\Sigma)$, and is such that $\theta(\Sigma)$ is a Cauchy surface for M .

On the other hand, the transformation properties of classical and quantum field theories under conformal rescalings of the metric have led, over the years, to many deep developments in mathematics and theoretical physics, e.g. conformal-infinity techniques in general relativity, twistor methods for gravitation and Yang–Mills theory, the conformal-variation method in heat-kernel asymptotics, the discovery of conformal anomalies in quantum field theory. All these topics are quite relevant for the analysis of theories which possess a gauge freedom. As a first example, one may consider the simplest gauge theory, i.e. vacuum Maxwell theory in four dimensions in the absence of sources. At the classical level, the operator acting on the potential A_b is found to be

$$P_a{}^b = -\delta_a{}^b \square + R_a{}^b + \nabla_a \nabla^b, \quad (2)$$

where ∇ is the Levi–Civita connection on space-time, $\square \equiv g^{ab}\nabla_a\nabla_b$, and R^{ab} is the Ricci tensor. Thus, the supplementary (or gauge) condition of the Lorenz type, i.e.

$$\nabla^b A_b = 0 \quad (3.a)$$

is of crucial importance to obtain a wave equation for A_b . The drawback of Eq. (3.a), however, is that it is not preserved under conformal rescalings of the metric:

$$\widehat{g}_{ab} = \Omega^2 g_{ab}, \quad \widehat{g}^{ab} = \Omega^{-2} g^{ab}, \quad (4)$$

whereas the Maxwell equations

$$\nabla^b F_{ab} = 0 \quad (5)$$

are invariant under the rescalings (4). This remark was the starting point of the investigation by Eastwood and Singer,² who found that a conformally invariant supplementary condition may be imposed, i.e.

$$\nabla_b \left[\left(\nabla^b \nabla^a - 2R^{ab} + \frac{2}{3} R g^{ab} \right) A_a \right] = 0. \quad (6.a)$$

As is clear from Eq. (6.a), conformal invariance is achieved at the price of introducing third-order derivatives of the potential. In flat backgrounds, such a condition reduces to

$$\square \nabla^b A_b = 0. \quad (7)$$

Of course, all solutions of the Lorenz gauge are also solutions of Eq. (7), whereas the converse does not hold.

Leaving aside the severe technical problems resulting from the attempt to quantize in the Eastwood–Singer gauge,³ we are now interested in understanding the key features of the counterpart for Einstein’s theory of general relativity. In other words, although the vacuum Einstein equations

$$R_{ab} - \frac{1}{2} g_{ab} R = 0 \quad (8)$$

are not invariant under the conformal rescalings (4), we would like to see whether the geometric structures leading to Eq. (6.a) admit a non-trivial generalization to Einstein’s theory, so that a conformally invariant supplementary condition with a higher order operator may be found as well. For this purpose, we re-express Eqs. (3.a) and (6.a) in the form

$$g^{ab} \nabla_a A_b = 0, \quad (3.b)$$

$$\begin{aligned} & g^{ab} \nabla_a \nabla_b \nabla^c A_c + \left[\nabla_b \left(-2R^{ba} + \frac{2}{3} R g^{ba} \right) \right] A_a \\ & + \left(-2R^{ba} + \frac{2}{3} R g^{ba} \right) \nabla_b A_a = 0. \end{aligned} \quad (6.b)$$

Eq. (3.b) involves the space-time metric in its contravariant form, which is also the metric on the bundle of 1-forms on M . In Einstein's theory, one deals instead with the vector bundle of symmetric rank-two tensors on space-time with DeWitt supermetric

$$E^{abcd} \equiv \frac{1}{2} \left(g^{ac} g^{bd} + g^{ad} g^{bc} + \alpha g^{ab} g^{cd} \right), \quad (9)$$

α being a real parameter different from $-\frac{2}{m}$, where m is the dimension of space-time (this restriction on α is necessary to make sure that the metric E^{abcd} has an inverse). One is thus led to replace Eq. (3.b) with the de Donder gauge

$$W^a \equiv E^{abcd} \nabla_b h_{cd} = 0. \quad (10)$$

Hereafter, h_{ab} denotes metric perturbations, since we are interested in linearized general relativity. The supplementary condition (10) is not invariant under conformal rescalings, but the expression of the Eastwood–Singer gauge in the form (6.b) suggests considering as a ‘candidate’ for a conformally invariant gauge involving a higher-order operator the equation

$$E^{abcd} \nabla_a \nabla_b \nabla_c \nabla_d W^e + \left[\left(\nabla_p T^{pebc} \right) + T^{pebc} \nabla_p \right] h_{bc} = 0. \quad (11)$$

More precisely, Eq. (11) is obtained from Eq. (6.b) by applying the replacement prescriptions

$$g^{ab} \rightarrow E^{abcd},$$

$$A_b \rightarrow h_{ab},$$

$$\nabla^b A_b \rightarrow W^e,$$

with T^{pebc} a rank-four tensor field obtained from the Riemann tensor, the Ricci tensor, the trace of Ricci and the metric. In other words, T^{pebc} is expected to involve all possible contributions of the kind R^{pebc} , $R^{pe} g^{bc}$, $R g^{pe} g^{bc}$, assuming that it should be linear in the curvature.

When a supplementary (or gauge) condition is imposed in a theory with gauge freedom, one of the first problems is to make sure that such a condition is preserved under the action of the gauge symmetry. More precisely, either the gauge is originally satisfied, and hence also the gauge-equivalent field configuration should fulfill the condition, or the gauge is not originally satisfied, but one wants to prove that, after performing a gauge transformation, it is always possible to fulfill the supplementary condition, eventually. The latter problem is the most general, and has a well known counterpart already for Maxwell theory.

For linearized classical general relativity in the family of gauges described by Eq. (11), the gauge symmetry remains the request of invariance under infinitesimal diffeomorphisms. Their effect on metric perturbations is given by

$$\varphi h_{ab} \equiv h_{ab} + (L_\varphi h)_{ab} = h_{ab} + \nabla_{(a} \varphi_{b)}. \quad (12)$$

For some smooth metric perturbation one might indeed have (cf. Eq. (11))

$$E^{abcd} \nabla_a \nabla_b \nabla_c \nabla_d W^e(h) + \left[(\nabla_p T^{pebc}) + T^{pebc} \nabla_p \right] h_{bc} \neq 0. \quad (13)$$

It is necessary to prove that one can, nevertheless, achieve the condition

$$E^{abcd} \nabla_a \nabla_b \nabla_c \nabla_d W^e(\varphi h) + \left[(\nabla_p T^{pebc}) + T^{pebc} \nabla_p \right] \varphi h_{bc} = 0. \quad (14)$$

Equation (14) is conveniently re-expressed in a form where the left-hand side involves a differential operator acting on the 1-form φ_q , and the right-hand side depends only on metric perturbations, their covariant derivatives and the Riemann curvature. Explicitly, one finds

$$P_e{}^q \varphi_q = -F_e, \quad (15)$$

where (hereafter h is the trace $g^{ab}h_{ab}$)

$$\begin{aligned} P_e{}^q \equiv & \left(\nabla^{(c} \nabla^{d)} \nabla_c \nabla_d + \frac{\alpha}{2} \square^2 \right) (\delta_e{}^q \square + \nabla^q \nabla_e + \alpha \nabla_e \nabla^q) \\ & + 2T_e{}^{(bq)}{}_{;p} \nabla_b + 2T_e{}^{(bq)} \nabla_p \nabla_b, \end{aligned} \quad (16)$$

$$\begin{aligned} F_e \equiv & 2 \left(\nabla^{(c} \nabla^{d)} \nabla_c \nabla_d + \frac{\alpha}{2} \square^2 \right) \left(\nabla^q h_{qe} + \frac{\alpha}{2} \nabla_e h \right) \\ & + 2T_e{}^{bc}{}_{;p} h_{bc} + 2T_e{}^{bc} \nabla_p h_{bc}. \end{aligned} \quad (17)$$

Our original work in Ref. 4 has proved the following results:

- (i) The value $\alpha = -2$ in the DeWitt supermetric (9) is ruled out if one wants to be able to solve Eq. (15) for φ_q .

(ii) If $\alpha = -1$, the general solution of Eq. (15) in flat m -dimensional Euclidean space \mathbf{E}^m reads

$$\begin{aligned}\varphi_a(x) &= \Omega_a(x) + \int_{\mathbf{E}^m} G_a^b(x, y) w_b(y) dy \\ &+ \int_{\mathbf{E}^m} \int_{\mathbf{E}^m} G_a^b(x, y) G_b^c(y, z) v_c(z) dy dz \\ &+ 2(2\pi)^{-\frac{m}{2}} \int_{\mathbf{E}^m} |\xi|^{-6} \widetilde{F}_a(\xi) e^{i\xi \cdot x} d\xi,\end{aligned}\tag{18}$$

where Ω_a, w_a and v_a are harmonic 1-forms in \mathbf{E}^m , G_a^b is the Green kernel of the Laplacian acting on 1-forms, and $|\xi| \equiv \sqrt{\xi_a \xi^a}$. In the last integral in Eq. (18) the only poles of the integrand occur when

$$\xi_0 = \pm i \sqrt{\sum_{k=1}^{m-1} \xi_k \xi^k},$$

i.e. on the imaginary ξ_0 axis. Thus, integration on the real line for ξ_0 , and subsequent integration with respect to ξ_1, \dots, ξ_{m-1} , yields a well defined integral representation of φ_q .

(iii) On compact Riemannian manifolds (M, g) without boundary and with non-vanishing Riemann curvature, Eq. (15) can be turned into the integral equation

$$\begin{aligned}\varphi_e(x) &+ \int_M \mathcal{G}_e^p(x, y) \left(\mathcal{B}_p^r \varphi_r \right)(y) \sqrt{\det g(y)} dy \\ &+ \int_M \mathcal{G}_e^p(x, y) F_p(y) \sqrt{\det g(y)} dy = 0,\end{aligned}\tag{19}$$

where \mathcal{G}_e^p is the Green kernel of the operator

$$\mathcal{A}_e^q \equiv \left(\nabla^{(c} \nabla^{d)} \nabla_c \nabla_d + \frac{\alpha}{2} \square^2 \right) (\delta_e^q \square + \nabla^q \nabla_e + \alpha \nabla_e \nabla^q),\tag{20}$$

and

$$\mathcal{B}_e^q \equiv 2T_e^{p(bq)}{}_{;p} \nabla_b + 2T_e^{p(bq)} \nabla_p \nabla_b.\tag{21}$$

A recursive algorithm for the solution of Eq. (19) can be developed provided that \mathcal{B}_e^q is a symmetric elliptic operator, so that it admits a discrete spectral resolution with eigenvectors of class C^∞ . The ellipticity condition means that the leading symbol of \mathcal{B}_e^q is non-vanishing for $\xi \neq 0$, i.e.

$$-2T_e^{p(bq)} \xi_p \xi_b \neq 0 \text{ for } \xi \neq 0.\tag{22}$$

This condition receives contributions from the parts of T involving the Ricci tensor and the scalar curvature, but not from the Riemann tensor, which is antisymmetric in b and q . For a given choice of background with associated curvature and tensor T , the above condition provides a useful operational criterion to check the admissibility of our supplementary condition (11).

The unsolved problem of our investigation is how to choose, or determine, the form of the tensor field T^{pebc} in the supplementary condition (11). If one writes for T^{pebc} the most general combination of Riemann, Ricci, trace of Ricci and background metric, it remains very difficult to study the behaviour of Eq. (11) under conformal rescalings. For example, the term which is known explicitly in Eq. (11) reads

$$\begin{aligned} E^{abcd}\nabla_a\nabla_b\nabla_c\nabla_dW^e &= (2+\alpha)\square^2W^e \\ &+ 2\left[\left(\left(\nabla_hR^{de}\right)-\left(\nabla^eR^d_h\right)\right)\left(\nabla_dW^h\right)+\left(\nabla^hR\right)\left(\nabla_hW^e\right)\right] \\ &+ 2R^{ah}\nabla_a\nabla_hW^e + \frac{3}{2}R^e_{qab}R_h^{qab}W^h, \end{aligned} \quad (23)$$

and one has, under conformal rescalings, the well known transformation properties of Riemann and Ricci, jointly with

$$\widehat{W}^a = \Omega^{-4} \left[W^a + (m-2)h^{ar}Y_r - (1+\alpha)\widehat{h}g^{ar}Y_r \right], \quad (24)$$

where $Y_r \equiv \nabla_r \log \Omega$, $\widehat{h} \equiv g^{cd}h_{cd}$. The next task is to check whether the conformal variation of the right-hand side of (23) compensates the conformal variation of $\nabla_p(T^{pebc}h_{bc})$ for a suitable form of T^{pebc} . It should also be stressed that the results (ii) and (iii) deal with the Riemannian rather than the Lorentzian case. The work in Ref. 4 has also performed the analysis in a Minkowskian background, but a curved Lorentzian background might lead to some novel features because it is then impossible to use the spectral theory of elliptic operators on manifolds.

The above results and open problems seem to suggest that new perspectives are in sight in the investigation of supplementary conditions in general relativity. They might have applications both in classical theory (linearized equations in gravitational wave theory, symmetry principles and their impact on gauge conditions), and in the attempts to quantize the gravitational field. In particular, the quantization via Euclidean path integrals requires adding to the Euclidean Einstein-Hilbert action I_{EH} (supplemented by a boundary term) the integral I_{GA} over M of $\chi^a\beta_{ab}\chi^b$, where χ^a is a gauge-averaging functional⁵ and β_{ab} is an invertible matrix. The sum of the integrals I_{EH} and I_{GA} is what we mean by full

Euclidean action for gravity (but there is, of course, also the ghost action.⁵) If χ^a contains fifth-order covariant derivatives of h_{ab} and curvature terms as we have proposed, it is not *a priori* obvious that the full Euclidean action remains unbounded from below.⁶ One might instead hope to combine ellipticity of the theory (now ruled by the leading symbol of a tenth-order differential operator resulting from $\chi^a \beta_{ab} \chi^b$) with the need to obtain a full Euclidean action for gravity which is bounded from below. For this purpose, only explicit calculations with a definite form of the tensor T^{pebc} can help to settle the issue.

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